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STRICTLY IRREDUCIBLE *-REPRESENTATIONS OF BANACH *-ALGEBRAS(1)

BY

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ABSTRACT. In this paper strictly irreducible *-representations of Banach *-algebras and the positive functionals associated with these representations are studied.

Introduction. Let A be a Banach *-algebra, and let $a \to \pi(a)$ be a representation of A on a Hilbert space $\mathcal{H}$. A subspace $\mathcal{K} \subset \mathcal{H}$ is π-invariant if $\pi(a)\mathcal{K} \subset \mathcal{K}$ for every $a \in A$. The representation π is irreducible if π is nonzero and the only closed π-invariant subspaces of $\mathcal{H}$ are $\mathcal{H}$ and $\{0\}$. π is strictly irreducible if π is nonzero and the only π-invariant subspaces of $\mathcal{H}$ are $\mathcal{H}$ and $\{0\}$. In the case where A is a B^*-algebra, R. V. Kadison proved the remarkable result that every irreducible *-representation of A is strictly irreducible [6, Theorem 1]. Aside from this theorem of Kadison, there are only a few minor isolated results concerning strictly irreducible *-representations of Banach *-algebras. In this paper we study strictly irreducible *-representations and certain positive functionals associated with these representations which we call strictly pure states (a positive functional α on A is a strictly pure state if α is a pure state and the *-representation of A determined by α is strictly irreducible). We give necessary and sufficient conditions that a pure state of A be strictly pure in §2. In §§3 and 4 some of the special properties of strictly pure states and strictly irreducible representations are presented. In §5 some examples of Banach *-algebras with the property that every irreducible *-representation is strictly irreducible are provided.

1. **Notation and preliminaries.** Throughout this paper A denotes a Banach *-algebra. A linear functional α on A is positive if $\alpha(a^*a) \geq 0$ for all $a \in A$. When α is a positive functional on A, let

$$M(\alpha) = \sup\left\{ \frac{|\alpha(a)|^2}{\alpha(a^*a)} \,\middle|\, a \in A,\ \alpha(a^*a) \neq 0 \right\}.$$

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The set of all positive functionals α on A with the properties $\alpha(a^*) = \overline{\alpha(a)}$ for all $a \in A$ and $M(\alpha) < +\infty$, we denote by $\mathcal{P}$. $\mathcal{P}_1$ is the set of all $\alpha \in \mathcal{P}$ with $M(\alpha) \leq 1$. Let A_h be the real linear space of hermitian elements of A. $\mathcal{P}_1$ is a convex subset of A_h^*, the dual space of A_h, and $\mathcal{P}_1$ is compact in the weak *-topology on A_h^* (see [4, Theorem (21.33), p. 328]). The extreme points of $\mathcal{P}_1$ are called pure states. For $\alpha \in \mathcal{P}$, the left kernel of α, denoted K_α, is the set of all $a \in A$ such that $\alpha(a^*a) = 0$. K_α is a closed left ideal of A. The quotient space $A - K_\alpha$ is a pre-Hilbert space in the inner-product $(a + K_\alpha, b + K_\alpha) = \alpha(b^*a)$. Let $\mathcal{H}_\alpha$ denote the Hilbert space which is the completion of this pre-Hilbert space. A *-representation $a \to \pi_\alpha(a)$ of A on $\mathcal{H}_\alpha$ is constructed by first defining $\pi_\alpha(a)(b + K_\alpha) = ab + K_\alpha$ for $b + K_\alpha \in A - K_\alpha$. Then $\pi_\alpha(a)$ is a bounded operator on $A - K_\alpha$ which extends uniquely to a bounded operator on $\mathcal{H}_\alpha$ (also denoted by $\pi_\alpha(a)$). For details of this construction see the proof of Theorem (21.24) in [4]. It is a well-known theorem that $\alpha \in \mathcal{P}$ is a pure state of A if and only if $M(\alpha) = 1$ and the *-representation π_α is irreducible on $\mathcal{H}_\alpha$ [4, Theorem (21.34), p. 328]. We define $\alpha \in \mathcal{P}$ to be a strictly pure state of A if α is a pure state of A and $a \to \pi_\alpha(a)$ is strictly irreducible on $\mathcal{H}_\alpha$.

When X is a normed linear space with norm $\|\cdot\|$ and Y is a closed subspace of X, then the quotient norm $\|\cdot\|_q$ on the quotient space $X - Y$ is defined as usual by

$$\|x + Y\|_q = \inf\{\|x - y\| \mid y \in Y\}.$$

$\mathcal{H}$ is always a Hilbert space and $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded operators on $\mathcal{H}$.

2. Necessary and sufficient conditions for a pure state to be strictly pure.

When $\alpha \in \mathcal{P}$, the quotient space $A - K_\alpha$ is an inner product space with inner product defined by $(a + K_\alpha, b + K_\alpha) = \alpha(b^*a)$. Let $|a + K_\alpha|_2 = (a + K_\alpha, a + K_\alpha)^{1/2} = \alpha(a^*a)^{1/2}$. We prove that a pure state α of A is strictly pure if and only if $A - K_\alpha$ is complete in the norm $|a + K_\alpha|_2$.

Theorem 2.1. *Assume that A is a Banach *-algebra and that α is a pure state of A. Then α is a strictly pure state of A if and only if $A - K_\alpha$ is complete in the norm $|a + K_\alpha|_2 = \alpha(a^*a)^{1/2}$. Also when α is a strictly pure state of A, then K_α is a modular maximal left ideal of A.*

Proof. Assume first that α is a strictly pure state of A. By the construction of $\mathcal{H}_\alpha$, $A - K_\alpha$ is an invariant subspace for $\pi_\alpha(a)$ whenever $a \in A$. Then $\mathcal{H}_\alpha = A - K_\alpha$, so that $A - K_\alpha$ is complete in the norm $|a + K_\alpha|_2$.

Conversely assume that $A - K_\alpha$ is complete in this norm. We prove first that the two norms $|\cdot|_2$ and $\|\cdot\|_q$ are equivalent on $A - K_\alpha$. By the Closed Graph Theorem it suffices to prove that $\|\cdot\|_q$ dominates $|\cdot|_2$. This is exactly the same as proving that the identity map $a + K_\alpha \to a + K_\alpha$ is a continuous map from $(A - K_\alpha, \|\cdot\|_q)$ onto $(A - K_\alpha, |\cdot|_2)$. Again using the Closed Graph Theorem, it suffices to show that this map is closed. Therefore assume that $\{a_n\} \subset A$, $a \in A$, $\|a_n + K_\alpha\|_q \to 0$, and $|(a_n - a) + K_\alpha|_2 \to 0$. Then there exists a sequence $\{k_n\} \subset K_\alpha$ such that $\|a_n + k_n\| \to 0$. Therefore $\|a^*a_n + a^*k_n\| \to 0$, and this implies $\alpha(a^*a_n) = \alpha(a^*a_n + a^*k_n) \to 0$. But also $|\alpha(a^*(a_n - a))| = |((a_n - a) + K_\alpha, a + K_\alpha)| \leq |(a_n - a) + K_\alpha|_2 |a + K_\alpha|_2 \to 0$. Therefore $\alpha(a^*a) = 0$, so that $a + K_\alpha = 0$.

Now define a functional $\bar{\alpha}$ on the Hilbert space $\mathcal{H}_\alpha = A - K_\alpha$ by $\bar{\alpha}(a + K_\alpha) = \alpha(a)$. Since K_α is contained in the null space of α, $\bar{\alpha}$ is well defined. Also,

$$\|\bar{\alpha}\|^2 = \sup\left\{\frac{|\alpha(a)|^2}{\alpha(a^*a)} \,\middle|\, a \in A,\ \alpha(a^*a) \neq 0\right\} = M(\alpha) = 1.$$

Since $A - K_\alpha$ is a Hilbert space, there exists $v \in A$ such that $\bar{\alpha}(a + K_\alpha) = (a + K_\alpha, v + K_\alpha) = \alpha(v^*a)$ for all $a \in A$. Therefore $\alpha(a) = \alpha(v^*a)$ for all $a \in A$. Given any $a \in A$,

$$\alpha((a(1 - v))^*(a(1 - v))) = \alpha(a^*a(1 - v)) - \alpha(v^*a^*a(1 - v)) = 0.$$

Therefore $A(1 - v) \subset K_\alpha$, so that K_α is a modular left ideal. Let K be a maximal left ideal of A such that $K_\alpha \subset K$. Set $M = \{b + K_\alpha \mid b \in K\}$. M is a proper π_α-invariant subspace of $\mathcal{H}_\alpha = A - K_\alpha$. Furthermore M is $\|\cdot\|_q$-closed. Therefore by the result of the previous paragraph, M is $|\cdot|_2$-closed. It follows that $K_\alpha = K$. Then since K_α is a maximal modular left ideal of A, $\pi_\alpha(A)$ acts strictly irreducibly on $\mathcal{H}_\alpha = A - K_\alpha$.

Every Banach *-algebra A has an algebra pseudonorm called the Gelfand-Naimark pseudonorm. We denote this pseudonorm by $|a|$, $a \in A$. This pseudonorm has the properties:

(1) $|a^*a| = |a|^2$ for all $a \in A$.

(2) $|\alpha(a)| \leq M(\alpha)|a|$ whenever $\alpha \in \mathcal{P}$, $a \in A$.

(3) The *-radical of A is the set of all $a \in A$ such that $|a| = 0$. See [8, p. 226] for the details of these results. We prove next that a pure state α of A is strictly pure if and only if $|a + K_\alpha|_q = \inf\{|a + k| \mid k \in K_\alpha\}$ is a complete norm on $A - K_\alpha$.

Theorem 2.2. *Let $|\cdot|$ denote the Gelfand-Naimark pseudonorm on A. Then a pure state α of A is strictly pure if and only if $|a + K_\alpha|_q$ is a complete norm on $A - K_\alpha$.*

Proof. For convenience we assume in the proof that A is reduced (i.e. the *-radical of A is 0). This assumption can be made with no loss of generality. In this case $|\cdot|$ is a norm on A with the B^*-property by (1) and (3) above. Let B denote the B^*-algebra which is the completion of A in the norm $|\cdot|$. Let α

be a pure state of A. By (2) above α is $|\cdot|$-continuous. Therefore α has a unique extension $\tilde{\alpha}$ to B. It is easy to verify that $\tilde{\alpha}$ is a pure state of B.

Now assume that α is a strictly pure state of A. Let $\mathrm{cl}(K_\alpha)$ denote the $|\cdot|$-closure of K_α in B. If $\mathrm{cl}(K_\alpha) \neq K_{\tilde{\alpha}}$, then by [8, Theorem (4.9.8), p. 251] there exists a pure state $\tilde{\beta}$ of B with $\mathrm{cl}(K_\alpha) \subset K_{\tilde{\beta}}$ and $\tilde{\alpha} \neq \tilde{\beta}$. Let β be the restriction of $\tilde{\beta}$ to A. $K_\alpha \subset K_\beta$ and therefore $K_\alpha = K_\beta$. By a result we prove in the next section, Theorem 3.2, it follows that $\alpha = \beta$. But then $\tilde{\alpha} = \tilde{\beta}$, a contradiction. Therefore $\mathrm{cl}(K_\alpha) = K_{\tilde{\alpha}}$. By Kadison's theorem $\tilde{\alpha}$ is a strictly pure state of B. Then as noted in Theorem 2.1 there exists $M > 0$ such that

$$M\tilde{\alpha}(b^*b)^{1/2} \geq |b + K_{\tilde{\alpha}}|_q \quad \text{for all } b \in B.$$

Also using (2) above we have, for $a \in A$, $k \in K_\alpha$,

$$|a + K_\alpha|_2 = \alpha((a+k)^*(a+k))^{1/2} \leq |(a+k)^*(a+k)|^{1/2} = |a+k|.$$

Therefore $|a + K_\alpha|_2 \leq |a + K_\alpha|_q$. Then for all $a \in A$,

$$M|a + K_\alpha|_2 = M\tilde{\alpha}(a^*a)^{1/2} \geq |a + K_{\tilde{\alpha}}|_q = |a + K_\alpha|_q \geq |a + K_\alpha|_2.$$

The norm $|a + K_\alpha|_2$ is complete on $A - K_\alpha$ by Theorem 2.1. Therefore $|a + K_\alpha|_q$ is a complete norm on $A - K_\alpha$.

Conversely assume that $|a + K_\alpha|_q$ is a complete norm on $A - K_\alpha$. Given $b \in K_{\tilde{\alpha}}$, choose $\{b_n\} \subset A$ such that $|b_n - b| \to 0$. Then $|(b_n - b_m) + K_\alpha|_q \to 0$ as $n, m \to +\infty$. Therefore there exists $a \in A$ such that $|(b_n - a) + K_\alpha|_q \to 0$. Choose $\{k_n\} \subset K_\alpha$ such that $|b_n - a + k_n| \to 0$. Then $|b - a + k_n| \to 0$, so that $b - a \in \mathrm{cl}(K_\alpha)$. It follows that $a^*b - a^*a \in \mathrm{cl}(K_\alpha)$, and therefore that $\tilde{\alpha}(a^*b - a^*a) = 0$. But $\tilde{\alpha}(a^*b) = 0$, since $b \in K_{\tilde{\alpha}}$. Then $\alpha(a^*a) = 0$, so that $a \in K_\alpha$. Therefore $b \in \mathrm{cl}(K_\alpha)$. We have now shown that $K_{\tilde{\alpha}} = \mathrm{cl}(K_\alpha)$. We have $|a + K_\alpha|_q \geq |a + K_\alpha|_2$ for all $a \in A$ just as before. By Kadison's theorem $\tilde{\alpha}$ is a strictly pure state of B. Then by Theorem 2.1 there exists $m > 0$ such that $|b + K_{\tilde{\alpha}}|_2 \geq m|b + K_{\tilde{\alpha}}|_q$ for all $b \in B$. Therefore for all $a \in A$,

$$|a + K_\alpha|_q \geq |a + K_\alpha|_2 = |a + K_{\tilde{\alpha}}|_2 \geq m|a + K_{\tilde{\alpha}}|_q = m|a + K_\alpha|_q.$$

It follows that $|a + K_\alpha|_2$ is a complete norm on $A - K_\alpha$, and therefore α is strictly pure by Theorem 2.1.

3. **Results concerning strictly pure states and strictly irreducible representations.** The relationship between a pure state and its left kernel has never been fully explored in a general Banach *-algebra. In fact to our knowledge none of the following questions have been answered when A is a Banach algebra with hermitian involution.

Question 1. If α is a pure state of A, is K_α a maximal left ideal of A?

Question 2. If α and β are pure states of A and $K_\alpha = K_\beta$, does $\alpha = \beta$?

Question 3. If $\alpha \in \mathcal{P}$, $M(\alpha) = 1$, and K_α is a maximal left ideal of A, is α a pure state of A?

We add to this list another closely related question.

Question 4. If $a \rightarrow \pi(a)$ and $a \rightarrow \gamma(a)$ are two algebraically equivalent irreducible *-representations of A on respective Hilbert spaces, are π and γ necessarily unitarily equivalent?

The answer to all these questions is affirmative when A is a B^*-algebra. In this section we deal with special cases of these questions. To begin with, Theorem 2.1 states that when α is a strictly pure state of A, then K_α is a modular maximal left ideal of A. This answers Question 1 in the case when α is strictly pure.

Next we prove a result which easily settles Question 2 if α or β is strictly pure. Kadison proves in [6] that when α is a pure state of a B^*-algebra, then $\mathfrak{N}(\alpha) = K_\alpha + K_\alpha^*$ where $\mathfrak{N}(\alpha)$ is the null space of α. We have the following generalization.

Proposition 3.1. *If α is a strictly pure state of A, then* $\mathfrak{N}(\alpha) = \overline{K_\alpha + K_\alpha^*}$.

Proof. Since $M(\alpha) = 1$, then $|\alpha(a)|^2 \leq \alpha(a^*a)$ for all $a \in A$. Therefore $K_\alpha \subset \mathfrak{N}(\alpha)$, and it follows that $K_\alpha + K_\alpha^* \subset \mathfrak{N}(\alpha)$. Now we prove the reverse inclusion. By Theorem 2.1, K_α is a modular left ideal of A. Therefore there exists $u \in A$ such that $A(1-u) \subset K_\alpha$. When $a \in \mathfrak{N}(\alpha)$, then $a^* \in \mathfrak{N}(\alpha)$, and $(u + K_\alpha, a + K_\alpha) = \alpha(a^*u) = \alpha(a^*u - a^*) = 0$. Thus $u + K_\alpha$ is orthogonal to $a + K_\alpha$ in $A - K_\alpha = \mathcal{H}_\alpha$. $\pi_\alpha(A)$ is a *-subalgebra of $\mathcal{B}(\mathcal{H}_\alpha)$ which acts strictly irreducibly on $\mathcal{H}_\alpha$. Let B be the closure of $\pi_\alpha(A)$ in the operator norm. By the transitivity theorem [3, Théorème (2.8.3)] there exists $T \in B$, $T = T^*$, such that $T(u + K_\alpha) = 0$ and $T(a + K_\alpha) = a + K_\alpha$. Then there exists $\{v_n\} \subset A$ such that $v_n = v_n^*$ for all n and $|\pi_\alpha(v_n) - T| \rightarrow 0$ where $|\cdot|$ denotes the operator norm. Therefore $|v_n(u + K_\alpha)|_2 \rightarrow 0$ and $|(v_n a - a) + K_\alpha|_2 \rightarrow 0$. Also $v_n = v_n(1-u) + v_n u$ and $v_n(1-u) \in K_\alpha$ for all n. Then $|v_n + K_\alpha|_2 \rightarrow 0$, and finally $|a^*v_n + K_\alpha|_2 \rightarrow 0$. From the proof of Theorem 2.1 it follows that $\|a^*v_n + K_\alpha\|_q \rightarrow 0$ and $\|(v_n a - a) + K_\alpha\|_q \rightarrow 0$. Assume for the moment the * is continuous on A. There exists $\{k_n\}, \{j_n\} \subset K_\alpha$ such that $\|a^*v_n - k_n\| \rightarrow 0$ and $\|(a - v_n a) - j_n\| \rightarrow 0$. Then $\|a - (j_n + k_n^*)\| \leq \|v_n a - k_n^*\| + \|(a - v_n a) - j_n\| \rightarrow 0$. Therefore in this case $\mathfrak{N}(\alpha) = \overline{K_\alpha + K_\alpha^*}$. In the general case, let P_α be the kernel of the representation π_α. A/P_α is a semisimple Banach *-algebra. Note that $P_\alpha \subset K_\alpha \cap K_\alpha^*$. Define α_0 on $a + P_\alpha \in A/P_\alpha$ by $\alpha_0(a + P_\alpha) = \alpha(a)$. Then α_0 is a strictly pure state of A/P_α. By Johnson's theorem [5, Theorem 2] the involution on A/P_α is continuous. Therefore $\mathfrak{N}(\alpha_0) = \overline{K_{\alpha_0} + K_{\alpha_0}^*}$ by our previous argument. Then when $a \in \mathfrak{N}(\alpha)$, there exists $\{k_n\}, \{j_n\} \subset K_\alpha$ such that $\|(a - (k_n + j_n^*)) + P_\alpha\|_q \rightarrow 0$. Then there exists

$\{p_n\} \subset P_\alpha$ such that $\|a - (k_n + p_n + j_n^*)\| \longrightarrow 0$. This proves the proposition.

We are now in a position to answer Question 2 affirmatively when α is assumed to be a strictly pure state of A.

Theorem 3.2. *Let α be a strictly pure state of A. Assume that $\beta \in \mathcal{P}$, $M(\beta) = 1$, and $K_\alpha = K_\beta$. Then $\alpha = \beta$.*

Proof. $K_\beta + K_\beta^* \subset \mathfrak{N}(\beta)$. Therefore

$$\mathfrak{N}(\alpha) = \overline{K_\alpha + K_\alpha^*} = \overline{K_\beta + K_\beta^*} \subset \mathfrak{N}(\beta).$$

It follows that there is a scalar $\lambda > 0$ such that $\alpha = \lambda\beta$. Then $1 = M(\alpha) = \lambda M(\beta) = \lambda$.

The next theorem answers Question 4 in a special case.

Theorem 3.3. *Assume that $\mathfrak{H}$ and $\mathfrak{K}$ are Hilbert spaces, and $a \longrightarrow \pi(a)$ and $a \longrightarrow \gamma(a)$ are strictly irreducible $*$-representations of A on $\mathfrak{H}$ and $\mathfrak{K}$ respectively. Then if π and γ are algebraically equivalent, then π and γ are unitarily equivalent.*

Proof. By hypothesis there exists a linear operator V which maps $\mathfrak{K}$ in a one-to-one manner onto $\mathfrak{H}$ with the property that $V^{-1}\pi(a)V = \gamma(a)$ for all $a \in A$. Take $\xi \in \mathfrak{K}$ with $\|\xi\| = 1$, and set $\alpha(a) = (\gamma(a)\xi, \xi)$ for $a \in A$. By [8, Lemma (4.5.8), p. 217] the representation γ is unitarily equivalent to π_α on $\mathfrak{H}_\alpha$. Also $M(\alpha) = \|\xi\|^2 = 1$ by [4, Theorem (21.25), p. 323]. Then α is a strictly pure state of A by [4, Theorem (21.34), p. 328]. Now set $\eta = V(\xi)/\|V(\xi)\|$. Define $\beta(a) = (\pi(a)\eta, \eta)$ for all $a \in A$. By the same argument as just applied to α, β is a strictly pure state of A, and π_β is unitarily equivalent to π. Then

$$\begin{aligned} a \in K_\alpha &\Leftrightarrow \gamma(a)\xi = 0 \Leftrightarrow \gamma(a)(V^{-1}(\eta)) = 0 \\ &\Leftrightarrow V^{-1}(\pi(a)(\eta)) = 0 \Leftrightarrow \pi(a)\eta = 0 \Leftrightarrow a \in K_\beta. \end{aligned}$$

Thus $K_\alpha = K_\beta$, and it follows from Theorem 3.2 that $\alpha = \beta$. Then $\pi_\alpha = \pi_\beta$, so that π and γ are unitarily equivalent.

To conclude this section we consider an answer to Question 3 when A has a very special property. We hypothesize that every maximal left ideal of A is the left kernel of a strictly pure state of A. In this case assume that α is as in Question 3, that is, $\alpha \in \mathcal{P}$, $M(\alpha) = 1$, and K_α is a maximal left ideal of A. By the special hypothesis on A there is a strictly pure state β of A such that $K_\beta = K_\alpha$ Then by Theorem 3.2, $\alpha = \beta$. We state this result as a proposition.

Proposition 3.4. *Assume that every maximal (modular) left ideal of A is the left kernel of a strictly pure state of A. When $\alpha \in \mathcal{P}$, $M(\alpha) = 1$, and K_α is a maximal (modular) left ideal of A, then α is a strictly pure state of A.*

4. **Irreducible representations which are similar to *-representations.** Let $a \longrightarrow \pi(a)$ be a strictly irreducible representation (but not necessarily a *-representation) of A on a Hilbert space $\mathcal{H}$. If $\xi \in \mathcal{H}$, $\xi \neq 0$, then a straightforward algebraic argument proves that $K_\xi = \{a \in A \mid \pi(a)\xi = 0\}$ is a modular maximal left ideal of A. We show in the next theorem that when K_ξ is the left kernel of a strictly pure state α of A, then π is similar to a *-representation of A on $\mathcal{H}$.

Theorem 4.1. *Let $a \longrightarrow \pi(a)$ and K_ξ be as above. Assume that α is a strictly pure state of A with $K_\alpha = K_\xi$. Then there exists a *-representation $a \longrightarrow \rho(a)$ of A on $\mathcal{H}$ and a positive operator $V \in \mathcal{B}(\mathcal{H})$ such that, for all $a \in A$,*

$$\pi(a) = V^{-1}\rho(a)V.$$

Proof. Since π is strictly irreducible, $a \longrightarrow \pi(a)\xi$ is a linear map from A onto $\mathcal{H}$. We define a sesquilinear form $[\cdot\,,\cdot]$ on $\mathcal{H} \times \mathcal{H}$ by

$$[\pi(a)\xi,\ \pi(b)\xi] = \alpha(b^*a),$$

$a,\ b \in A$. Whenever $c \in K_\alpha$ and $d \in A$, then $\alpha(d^*c) = 0$. This implies that $[\cdot\,,\cdot]$ is well defined.

Next we prove that $[\cdot,\ \cdot]$ is a bounded form. By a theorem of B. E. Johnson [5, Theorem 1, p. 537] π is a continuous map of A into $\mathcal{B}(\mathcal{H})$. If $k \in K_\xi$, $a \in A$,

$$\|\pi(a)\xi\| = \|\pi(a+k)\xi\| \le \|\pi\|\,\|\xi\|\,\|a+k\|.$$

Therefore for any $a \in A$,

$$\|\pi(a)\xi\| \le \|\pi\|\,\|\xi\|\,\|a + K_\xi\|_q. \tag{1}$$

Then by the Closed Graph Theorem there exists $N > 0$ such that, for all $a \in A$,

$$\|a + K_\xi\|_q \le N\|\pi(a)\xi\|. \tag{2}$$

As shown in the proof of Theorem 2.1, the norms $|\cdot|_2$ and $\|\cdot\|_q$ are equivalent on $A - K_\alpha$. In particular there exists $J > 0$ such that $|a + K_\alpha|_2 \le J\|a + K_\alpha\|_q$ for all $a \in A$. Therefore for all $a,\ b \in A$,

$$|\alpha(b^*a)| = |(a + K_\alpha,\ b + K_\alpha)| \le J^2\|a + K_\alpha\|_q\|b + K_\alpha\|_q. \tag{3}$$

Then combining (2) and (3) we have

$$|[\pi(a)\xi,\ \pi(b)\xi]| = |\alpha(b^*a)| \le J^2\|a + K_\alpha\|_q\,\|b + K_\alpha\|_q \le J^2N^2\|\pi(a)\xi\|\,\|\pi(b)\xi\|.$$

This proves that $[\cdot,\ \cdot]$ is bounded on $\mathcal{H} \times \mathcal{H}$.

The form $[\cdot,\ \cdot]$ is a symmetric, positive definite, bounded sesquilinear form on $\mathcal{H} \times \mathcal{H}$. Therefore there exists an operator $U \in \mathcal{B}(\mathcal{H})$ such that $U = U^*$, $U \ge 0$, and $[\phi, \psi] = (U\phi, \psi)$, when $\phi, \psi \in \mathcal{H}$.

By (3), for all $a \in A$,

$$(|a + K_\alpha|_2)^2 = \alpha(a^*a) \le J^2(\|a + K_\alpha\|_q)^2.$$

By the proof of Theorem 2.1 there exists $P > 0$ such that, for all $a \in A$,

$$\|a + K_\alpha\|_q \leq P|a + K_\alpha|_2.$$

Given $a \in A$, set $\psi = \pi(a)\xi$. Then by (1),

$$\|\psi\| = \|\pi(a)\xi\| \leq \|\pi\| \|\xi\| \|a + K_\alpha\|_q.$$

Set $M = \|\pi\| \|\xi\|P$. Then

$$\|\psi\|^2 \leq M^2(|a + K_\alpha|_2)^2 = M^2\alpha(a^*a) = M^2[\psi, \psi].$$

Therefore

$$\|\psi\|^2 \leq M^2[\psi, \psi] = M^2 (U\psi, \psi) \leq M^2\|U\psi\| \|\psi\|.$$

Finally $\|\psi\| \leq M^2\|U\psi\|$, and this proves that $U^{-1} \in \mathfrak{B}(\mathfrak{H})$.

Now set $V = U^{1/2}$. Then $[\phi, \psi] = (V\phi, V\psi)$ for all $\phi, \psi \in \mathfrak{H}$. Let $\rho(a) = V\pi(a)V^{-1}$, $a \in A$. Given $\psi_1, \psi_2 \in \mathfrak{H}$, there exists $\phi_1, \phi_2 \in \mathfrak{H}$ and $a_1, a_2 \in A$ such that

$$\psi_i = V\phi_i \text{ and } \phi_i = \pi(a_i)\xi, \qquad i = 1, 2.$$

Then

$$\begin{aligned}(\rho(a)\psi_1, \psi_2) &= (V\pi(a)V^{-1}V\phi_1, V\phi_2)\\ &= [\pi(a)\phi_1, \phi_2] = [\pi(a)\pi(a_1)\xi, \pi(a_2)\xi]\\ &= \alpha(a_2^*(aa_1)) = \alpha((a^*a_2)^*a_1)\\ &= [\pi(a_1)\xi, \pi(a^*)\pi(a_2)\xi] = [\phi_1, \pi(a^*)\phi_2]\\ &= (V\phi_1, V\pi(a^*)\phi_2) = (V\phi_1, V\pi(a^*)V^{-1}V\phi_2) = (\psi_1, \rho(a^*)\psi_2).\end{aligned}$$

Therefore $\rho(a^*) = \rho(a)^*$ for all $a \in A$ which completes the proof of the theorem.

Corollary 4.2. *Assume that every modular maximal left ideal of A is the left kernel of a strictly pure state of A. Let $a \rightarrow \pi(a)$ be a strictly irreducible representation of A on a Hilbert space $\mathfrak{H}$. Then there exists a $*$-representation $a \rightarrow \rho(a)$ of A on $\mathfrak{H}$ and a positive operator $V \in \mathfrak{B}(\mathfrak{H})$ such that, for all $a \in A$,*

$$\pi(a) = V^{-1}\rho(a)V.$$

5. **Some examples.** When A is B^*-algebra, then A has the following two properties:

(I) Every pure state of A is strictly pure.

(II) Every modular maximal left ideal of A is the left kernel of a strictly pure state of A.

Also when G is a compact topological group and $1 \leq p < +\infty$, then

$A = L^p(G)$ (or $C(G)$, the continuous functions on G) has properties (I) and (II). Here the multiplication is, as usual, convolution. All the irreducible *-representations of A in this case are finite dimensional. In this section we present two examples of algebras which have properties (I) and (II), but which are not in general B^*-algebras, and which need not in general have any finite dimentional *-representations.

Example 5.1. Let A be a Banach algebra which is also a dense *-ideal in a B^*-algebra B. Any full Hilbert algebra is a particular example of such a Banach algebra; see [1].

Assume that $a \longrightarrow \pi(a)$ is an irreducible *-representation of A on a Hilbert space $\mathcal{H}$. Then as shown in [1, Proposition 4.1] π extends uniquely to a *-representation $b \longrightarrow \tilde{\pi}(b)$ of B on $\mathcal{H}$. Therefore by Kadison's theorem $\tilde{\pi}(B)$ acts strictly irreducibly on $\mathcal{H}$. Since A is a dense ideal of B, $\pi(A) = \tilde{\pi}(A)$ is a nonzero ideal in $\tilde{\pi}(B)$. Given $\xi \in \mathcal{H}$, $\pi(A)\xi$ is a $\tilde{\pi}(B)$-invariant subspace of $\mathcal{H}$. Therefore $\pi(A)\xi = \mathcal{H}$, so that $a \longrightarrow \pi(a)$ is strictly irreducible on $\mathcal{H}$. It follows that A has property (I).

Now assume that M is a modular maximal left ideal of A. Then there exists $u \in A$ such that $A(1-u) \subset M$. Let $N = \{b \in B \mid bu \in M\}$. N is a left ideal of B and $M = N \cap A$. Furthermore if $b \in B$, $b(1-u)u = bu(1-u) \in M$ since $bu \in A$. Therefore N is a proper modular left ideal of B. By [8, Theorem (4.9.8), p. 251] there exists $\tilde{\alpha}$ a pure state of B with $N \subset K_{\tilde{\alpha}}$. Then $M = K_{\tilde{\alpha}} \cap A$. It follows that α, the restriction of $\tilde{\alpha}$ to A, is a strictly pure state of A with $K_\alpha = M$. We have shown that A has property (II).

Example 5.2. Assume that Ω is a compact Hausdorff space and B is a B^*-algebra with identity e. Let $C(\Omega, B)$ be the algebra of all continuous B-valued functions on Ω. $C(\Omega, B)$ is a B^*-algebra with identity. Assume that A is a Banach algebra which is a *-subalgebra of $C(\Omega, B)$ containing the identity. We also assume that A has the properties:

(1) Given $\omega \in \Omega$ and $b \in B$, there exists $f \in A$ such that $f(\omega) = b$.

(2) $f \in A$ is left invertible in A if and only if $f(\omega)$ is invertible in B for all $\omega \in \Omega$.

We mention a specific example of such an algebra A: Let Ω be the interval $[0, 2\pi]$ with 0 and 2π identified and with the usual topology. Let B be any B^*-algebra with identity. We define A to be the algebra of all functions of the form

$$f(t) = \sum_{n=-\infty}^{+\infty} a_n e^{int}$$

where $t \in \Omega$ and $\{a_n\}$ is any sequence in B such that $\sum_{n=-\infty}^{+\infty} \|a_n\| < +\infty$. When $f(t) = \sum_{n=-\infty}^{+\infty} a_n e^{int}$, let $\|f\| = \sum_{n=-\infty}^{+\infty} \|a_n\|$. The algebra A is discussed by

Bochner and Phillips in [2]. That A has property (2) above is the assertion of [2, Theorem 1, p. 409]. The rest of the required properties of A are easily verified.

Now assume that A is any Banach *-subalgebra of $C(\Omega, B)$ which contains the identity and satisfies (1) and (2). When $\omega \in \Omega$ and N is a maximal left ideal of B, we define

$$K(\omega, N) = \{f \in A \mid f(\omega) \in N\}.$$

It is not difficult to see that $K(\omega, N)$ is a maximal left ideal of A. We prove the converse of this. Assume that M is a maximal left ideal of A. For any $\omega \in \Omega$, $M(\omega) = \{f(\omega) \mid f \in M\}$ is a left ideal of B. Suppose that $M(\omega) = B$ for all $\omega \in \Omega$. Then for each $\omega \in \Omega$, we can choose a function $g_\omega \in M$ such that $g_\omega(\omega) = e$. Therefore there exists an open set U_ω in Ω such that $\omega \in U_\omega$ and $g_\omega(\gamma)$ is invertible in B for all $\gamma \in U_\omega$. Then $(g_\omega^* g_\omega)(\gamma)$ is invertible in B for all $\gamma \in U_\omega$. Choose a finite cover $U_{\omega_1}, \cdots, U_{\omega_n}$ for Ω. Set $f = \Sigma_{k=1}^n g_{\omega_k}^* g_{\omega_k} \in M$. When $h_k \in B$, $h_k \geq 0$, $1 \leq k \leq n$, and h_j is invertible for some j, then $h_1 + \cdots + h_n$ is invertible (this is easy to verify when the h_k are positive operators on a Hilbert space, since the lower bound of the numerical range of the sum $h_1 + \cdots + h_n$ is greater or equal to the lower bound of the numerical range of h_j). But then for all $\gamma \in \Omega$, $f(\gamma)$ is invertible in B. By (2), f is then invertible in A, which contradicts the fact that $f \in M$. It follows that for some $\omega \in \Omega$, $M(\omega)$ is a proper left ideal of B. Then there exists a maximal left ideal of B such that $M(\omega) \subset N$. Therefore $M \subset K(\omega, N)$, so that $M = K(\omega, N)$ by the assumption that M is maximal.

Given M a maximal left ideal of A, then as we have shown above $M = K(\omega, N)$ for some $\omega \in \Omega$ and some maximal left ideal N of B. Choose β a pure state of B such that $K_\beta = N$. Define α on A by $\alpha(f) = \beta(f(\omega))$, $f \in A$. Then $K_\alpha = K(\omega, N) = M$. It is easy to verify that the norm $|f + K_\alpha|_2 = \alpha(f^*f)^{1/2}$ is a complete norm on $A - K_\alpha$. Therefore α is a strictly pure state of A. This proves that A has property (II).

Now assume that α is a pure state of A. Then by [3, Lemma 2.10.1, p. 50] α has an extension to a pure state β of $C(\Omega, B)$. By [7, Corollary, p. 337] there exists a point $\omega \in \Omega$ and a maximal left ideal N of B such that $K_\beta = \{f \in C(\Omega, B) \mid f(\omega) \in N\}$. Therefore $K_\alpha = K_\beta \cap A = K(\omega, N)$. It follows that α is a strictly pure state of A by Proposition 3.4. Therefore A has property (I).

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